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GENERALIZED DIMENSION SUBGROUPS AND DERIVED FUNCTORS

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Dedicated to Donald S. Passman on his seventy-fifth birthday

Abstract. Every two-sided ideal $\mathfrak{a}$ in the integral group ring $\mathbb{Z}[G]$ of a group G determines a normal subgroup $G \cap (1+\mathfrak{a})$ of G. In this paper certain problems related to the identification of such subgroups, and their relationship with derived functors in the sense of Dold-Puppe, are discussed.

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Introduction

Let G be a group and $\mathbb{Z}[G]$ its integral group ring. Every two-sided ideal $\mathfrak{a}$ in the integral group ring $\mathbb{Z}[G]$ of a group G determines a normal subgroup $G\cap(1+\mathfrak{a})$ of G. The identification of such normal subgroups is a fundamental problem in the theory of group rings. On expressing the group G as a quotient of a free group F, the problem can clearly be viewed as an identification problem in the free group ring $\mathbb{Z}[F]$. Let $\mathfrak{g}$ denote the augmentation ideal of $\mathbb{Z}[G]$. In analogy with the case when $\mathfrak{a}$ is a power $\mathfrak{g}^n$, $n \geq 1$, of the augmentation ideal and the corresponding subgroup, denoted $D_n(G)$, is called the nth *dimension subgroup* of G, we call the normal subgroups $G \cap (1+\mathfrak{a}+\mathfrak{g}^n)$, $n \geq 1$, *generalized dimension subgroups*. For every group G and integer $n \geq 1$, it is easily seen that $D_n(G) \geq \gamma_n(G)$, the nth term in the lower central series of G. It is well-known that $D_3(G) = \gamma_3(G)$, or, equivalently, that $F\cap(1+\mathfrak{r}\mathbb{Z}[F]+\mathfrak{f}^3) = R\gamma_3(F)$ for every free presentation $G \cong F/R$. A relationship between generalized dimension subgroups and derived functors in the sense of Dold-Puppe [4] is noticed on considering the subgroup $F \cap (1+\mathfrak{r}\mathfrak{f}+\mathfrak{f}^3)$ (Theorem 12; [8], Theorem 3.3). Motivated by this observation, our aim in this paper is to explore further relations between generalized dimension subgroups and derived functors. We expect our approach will initiate further work leading to deeper understanding of generalized dimension subgroups.

Based on the investigations of Narain Gupta [7], we identify, in Section 1, generalized dimension subgroups $F \cap (1+\mathfrak{a}+\mathfrak{f}^4)$ for various two-sided ideals $\mathfrak{a}$ in a free group ring $\mathbb{Z}[F]$. In view of the fact that the dimension series $\{D_n(G)\}_{n\geq 1}$ and the lower central series $\{\gamma_n(G)\}_{n\geq1}$, in general, cease to be identical from

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$n = 4$ onwards, the identification of these subgroups becomes of particular interest. Typical of the cases that we consider is the identification of the subgroup $F \cap (1 + \mathfrak{f}\mathfrak{r}\mathfrak{f} + \mathfrak{f}^4)$ (Theorem 5). In Corollary 10 we show that $F \cap (1 + \mathfrak{r}^2\mathbb{Z}[F] + \mathfrak{f}^4) \leq R\gamma_4(F)$, thus answering Narain Gupta's Problem 6.9(a) in [7]. This result, however, leaves open for future investigation the interesting question whether $F \cap (1 + \mathfrak{r}^{n-1}\mathbb{Z}[F] + \mathfrak{f}^{n+1}) \leq R\gamma_{n+1}(F)$ for some $n \geq 4$.

In Section 2 we exhibit how some of the generalized dimension subgroups are related to derived functors. Our main result here is Theorem 13 relating $F \cap (1 + \mathfrak{f}\mathfrak{r}\mathfrak{f} + \mathfrak{f}^4)$ to the homology of the Eilenberg-MacLane space $K(G_{ab}, 2)$, where G_{ab} is the abelianization of the group $G \cong F/R$. More precisely, we prove that, if G_{ab} is 2-torsion-free, then $\frac{F\cap(1+\mathfrak{f}\mathfrak{r}\mathfrak{f}+\mathfrak{f}^4)}{[R,\, R,\, F]\gamma_4(F)} \cong L_1\,\mathrm{SP}^3(G_{ab})$, where $L_1\,\mathrm{SP}^3$ is the first derived functor of the third symmetric power functor, and there is a natural short exact sequence

$$\frac{F \cap (1 + \mathfrak{f}\mathfrak{r}\mathfrak{f} + \mathfrak{f}^4)}{[R,\, R,\, F]\gamma_4(F)} \hookrightarrow H_7K(G_{ab},\, 2) \twoheadrightarrow \mathrm{Tor}(G_{ab},\, \mathbb{Z}/3\mathbb{Z}). \tag{1}$$

Consequently, if G_{ab} is p-torsion-free for $p = 2,\ 3$, then there is a natural isomorphism

$$\frac{F \cap (1 + \mathfrak{f}\mathfrak{r}\mathfrak{f} + \mathfrak{f}^4)}{[R,\, R,\, F]\gamma_4(F)} \cong H_7K(G_{ab},\, 2). \tag{2}$$

We define a functor $\mathfrak{L}^3_s(A)$ on the category $\mathfrak{A}$ of abelian groups (see Section 2 for definition) and prove (Theorem 14) that if G is a group and F/R its free presentation, then

$$\frac{F \cap (1 + \mathfrak{r}^2\mathfrak{f} + \mathfrak{f}^4)}{\gamma_3(R)\gamma_4(F)} \cong L_2\mathfrak{L}^3_s(G_{ab}).$$

In Section 3 we study limits of functors from the category of free presentations of groups to the category of abelian groups, and show their connection with quadratic and cubic functors. Our main results on limits are Theorems 15 and 16 where we show, in particular, that $\varprojlim \frac{\gamma_2(F)}{\gamma_2(R)\gamma_3(F)}$, $\varprojlim \frac{\gamma_3(F)}{\gamma_3(R)\gamma_4(F)}$ and $\varprojlim \frac{\gamma_3(F)}{[\gamma_2(R),\, F]\gamma_4(F)}$ agree with certain derived functors. In particular, we show (see Theorem 16) that

$$\varprojlim \frac{\gamma_3(F)}{[R,\, R,\, F]\gamma_4(F)} \cong L_1\,\mathrm{SP}^3(G_{ab}).$$

For background results, we refer the reader to the monographs [7] and [13] and the article [9].

1. Generalized dimension subgroups

Let F be a free group with basis X. Then the augmentation ideal $\mathfrak{f}$ of the group ring $\mathbb{Z}[F]$ is a two-sided ideal which is free as a left (resp. right) $\mathbb{Z}[F]$-module with basis $\{x - 1 \,|\, x \in X\}$ ([6], p. 32). Thus every element $u \in \mathfrak{f}$ can be written uniquely as

$$u = \sum_{{}_xu\in\mathbb{Z}[F],\, x\in X} {}_xu(x-1) = \sum_{x\in X,\, u_x\in\mathbb{Z}[F]} (x-1)u_x.$$

We refer to ${}_xu$ (resp. u_x) as the *left* (resp. *right*) *partial derivative* of u with respect to the generator x.

Let F be a free group of finite rank with basis $\{x_1, \dots, x_m\}$, $e_1, e_2, \dots, e_m$ integers ≥ 0 satisfying $e_m|e_{m-1}|\dots|e_1$, and $S = \langle x_1^{e_1}, \dots, x_m^{e_m}, \gamma_2(F)\rangle$. For a two-sided ideal $\mathfrak{a}$ of the group ring $\mathbb{Z}[F]$, set

$$D(n, \mathfrak{a}) := F \cap (1 + \mathfrak{a} + \mathfrak{f}^n).$$

For $1 \leq i \leq m$, let

$$t(x_i, e_i) := \begin{cases} 1 + x_i + \dots + x_i^{e_i - 1}, \text{ if } e_i \geq 1, \\ 0, \text{ if } e_i = 0 \end{cases}$$

Recall the following result of Narain Gupta ([7], Theorem 3.2, p. 81).

Theorem 1. *For all $n \geq 1$, modulo $[\gamma_2(F), S]\gamma_{n+2}(F)$, the group $D(n+2, \mathfrak{fs})$ is generated by the elements*

$$[x_i, x_j]^{t(x_i, e_i)a_{ij}}, \ 1 \leq i < j \leq m,$$

where $a_{ij} = a_{ij}(x_j, \dots, x_m) \in \mathbb{Z}[F]$ and

$$t(x_i, e_i)a_{ij} \in t(x_j, e_j)\mathbb{Z}[F] + \mathfrak{s}\mathbb{Z}[F] + \mathfrak{f}^n.$$

The cases $n = 1, 2$ of the above Theorem yield the following identification.

Theorem 2. (i) *Modulo $\gamma_2(S)\gamma_3(F)$,*

$$D(3, \mathfrak{fs}) = \langle [x_i^{e_i}, x_j] \mid 1 \leq i < j \leq m\rangle.$$

(ii) *Modulo $\gamma_2(S)\gamma_4(F)$,*

$$D(4, \mathfrak{fs}) = \left\langle [x_i^{e_i}, x_j]^{a_{ij}} \mid 1 \leq i < j \leq m, \ e_j | \frac{e_i}{e_j}\binom{e_j}{2} a_{ij} \right\rangle.$$

Proof. We sketch the proof for (ii), the computations for (i) are similar and simpler.

It is straight-forward to check that all elements $[x_i^{e_i}, x_j]^{a_{ij}}$ with $1 \leq i < j \leq m$, $e_j|\frac{e_i}{e_j}\binom{e_j}{2}a_{ij}$, and the subgroup $\gamma_2(S)\gamma_4(F)$, are contained in $D(4, \mathfrak{fs})$.

Conversely, let $w \in D(4, \mathfrak{fs})$. Then, observe that modulo $\gamma_2(S)\gamma_4(F)$, $w = \prod_{i=1}^m w_i$, where

$$w_i = \prod_{j=i+1}^{m} [x_i, x_j]^{d_{ij}} \in D(4, \mathfrak{fs}), \tag{3}$$

with $d_{ij} = d_{ij}(x_i, \dots, x_m)$. From eq.(1) it follows that

$$d_{ij} \in t(x_i, e_i)\mathbb{Z}[F] + \mathfrak{f}^4 + \mathfrak{a}, \tag{4}$$

where $\mathfrak{a} = \mathbb{Z}[F](\gamma_2(F) - 1)$. Thus we have, modulo $\gamma_4(F)\gamma_2(S)$,

$$w_i = \prod_{j=i+1}^{m} [x_i^{e_i}, x_j]^{a_{ij}} \in D(4, \mathfrak{fs}), \tag{5}$$

where $a_{ij} = a_{ij}(x_i, \dots, x_m) \in \mathbb{Z}[F]$.

Eq.(3) is equivalent to saying that

$$t(x_i,\, e_i)\sum_{j=i+1}^{m}(x_j-1)a_{ij}\in \mathfrak{f}^3+\mathfrak{s}. \tag{6}$$

Recall that for $i<j,\ e_j|e_i$. Modulo $\mathfrak{f}^3+\mathfrak{s}$,

$$\begin{aligned} t(x_i,\, e_i)\sum_{j=i+1}^{m}(x_j-1)a_{ij} &= \{e_i+\binom{e_i}{2}(x_i-1)\}\sum_{j=i+1}^{m}(x_j-1)a_{ij} \\ &= \sum_{j=i+1}^{m} a_{ij}\frac{e_i}{e_j}e_j(x_j-1)+\binom{e_i}{2}(x_i-1)\sum_{j=i+1}^{m}(x_j-1)a_{ij} \\ &= -\sum_{j=i+1}^{m} a_{ij}\frac{e_i}{e_j}\binom{e_j}{2}(x_j-1)^2+\binom{e_i}{2}(x_i-1)\sum_{j=i+1}^{m}(x_j-1)a_{ij}. \end{aligned} \tag{7}$$

Since all terms in the last expression are in $\mathfrak{f}^2$, we can assume that the elements $a_{ij}\in\mathbb{Z}$.

Since $\mathrm{SP}^2(F/S)\simeq \mathfrak{f}^2/(\mathfrak{f}^3+\mathfrak{s})$, eq.(5) holds if and only if

$$e_j|\frac{e_i}{e_j}\binom{e_j}{2}a_{ij},\quad e_j|\binom{e_i}{2}a_{ij}. \tag{8}$$

Note that the latter divisibility requirement holds if the former does. Thus we have the claimed identification. □

Since $D(4,\,\mathfrak{fs})\le D(3,\,\mathfrak{fs})$ and $\gamma_4(F)\le\gamma_3(F)$, we have a natural map

$$\theta: D(4,\mathfrak{fs})/\gamma_2(S)\gamma_4(F)\to D(3,\mathfrak{fs})/\gamma_2(S)\gamma_3(F).$$

We claim that θ is a monomorphism. To this end, we need the following

Lemma 3. *$\gamma_3(F)\cap D(4,\,\mathfrak{fs})\le\gamma_4(F)\gamma_2(S)$.*

Proof. Let $w\in\gamma_3(F)\cap D(4,\,\mathfrak{fs})$. Then, modulo $\gamma_4(F)$, $w=\prod_{i=1}^m w_i$,

$$w_i=\prod_{j>i\le k}[[x_j,\,x_i],\,x_k]^{a_{ijk}}\in D(4,\,\mathfrak{fs}),\ \ a_{ijk}\in\mathbb{Z}.$$

We then have

$$\sum_{j>i\le k} a_{ijk}[x_j-1)(x_i-1)-(x_i-1)(x_j-1)](x_k-1)\in\mathfrak{f}^4+\mathfrak{fs}.$$

Comparing the right partial derivatives with respect to x_j, we have, for each $j>i$,

$$\sum_{k\ge i} a_{ijk}(x_i-1)(x_k-1)\in\mathfrak{f}^3+\mathfrak{s}\mathbb{Z}[F].$$

This is possible if and only if $e_k|a_{ijk}$ for each $k\ge i$. But then $[[x_j,\,x_i],\,x_k]^{a_ijk}\in\gamma_4(F)\gamma_2(S)$, and the assertion stands proved. □

Corollary 4. *The natural map*

$$\theta : D(4,\,\mathfrak{f}\mathfrak{s})/\gamma_2(S)\gamma_4(F) \to D(3,\mathfrak{f}\mathfrak{s})/\gamma_2(S)\gamma_3(F)$$

is a monomorphism, and the cokernel of θ is an elementary abelian 2-group.

Proof. Suppose $w \in D(4,\,\mathfrak{f}\mathfrak{s}) \cap (\gamma_2(S)\gamma_3(F))$. Then $w = uv,\ u \in \gamma_2(S),\ v \in \gamma_3(F)$. Since $u \in D(4,\,\mathfrak{f}\mathfrak{s})$, it follows that $v \in \gamma_3(F) \cap D(4,\,\mathfrak{f}\mathfrak{s})$, and therefore, by Lemma 3, $v \in \gamma_4(F)\gamma_2(S)$. Hence θ is a monomorphism.

The subgroup $D(3,\,\mathfrak{f}\mathfrak{s})$ is generated, modulo $\gamma_3(F)\gamma_2(S)$, by the elements $[x_j,\,x_i^{e_i}],\ j > i$. From the identification of $D(4,\,\mathfrak{f}\mathfrak{s})$ (Theorem 2), note that $[x_j,\,x_i^{e_i}]^2 \in D(4,\,\mathfrak{f}\mathfrak{s})$. It thus follows that coker θ, which is isomorphic to the quotient

$$D(3,\,\mathfrak{f}\mathfrak{s})/(D(4,\,\mathfrak{f}\mathfrak{s})\gamma_3(F),$$

is an elementary abelian 2-group. □

Theorem 5. *If F is a free group with ordered basis $\{x_1 < \cdots < x_m\}$, $S = \langle x_1^{e_1},\,\ldots,\,x^{e_m},\,\gamma_2(F)\rangle,\ e_m|e_{m-1}|\ldots|e_1,$ and R is a normal subgroup of F satisfying $R\gamma_2(F) = S$, then the following identifications hold:*

(i) $D(4,\,\mathfrak{f}\mathfrak{r}\mathfrak{f}))$ *is the subgroup generated, modulo $\gamma_4(F)$, by the elements of the type*

- $[[x_j,\,x_i],\,x_k]^{e_i},\ i,\,j,\,k \in \{1,\,\ldots\,,\,m\},\ j > i;$
- $[[x_j,\,x_i],\,x_j]^{e},\ j > i,\ e_j|e,\ e_i|2e.$

(ii) $D(4,\,\mathfrak{r}\mathfrak{f}\mathfrak{r})$ *is the subgroup generated, modulo $\gamma_4(F)$, by the elements of the type*

- $[[x_j,\,x_i],\,x_k]^{e_ie_k},\ i,\,j,\,k \in \{1,\,\ldots\,,\,m\},\ j > i;$
- $[[x_j,\,x_i],\,x_i]^{e},\ j > i,\ (e_je_i)|e,\ e_i^2|2e.$

(iii) $D(4,\,\mathfrak{r}^2\mathfrak{f} + \mathfrak{r}\mathfrak{f}\mathfrak{r} + \mathfrak{f}\mathfrak{r}^2) = [[F,R],R]\gamma_4(F).$

(iv) $D(4,\,\mathfrak{f}^2\mathfrak{r}{+}\mathfrak{r}\mathfrak{f}^2)$ *= the subgroup generated, modulo $\gamma_4(F)$, by the elements $[[x_j,\,x_i],\,x_k]^e$, where $j > i \le k$, $e_k|e$, if $k \neq i$, and when $k = i$, then $e_j|e$, and $e_i|2e$.*

Proof. (i) Note that $D(4,\,\mathfrak{f}\mathfrak{r}\mathfrak{f}) = D(4,\,\mathfrak{f}\mathfrak{s}\mathfrak{f}) \le \gamma_3(F)$. We first check that for $i,\,j,\,k \in \{1,\,\ldots\,,\,m\}$, the simple commutator

$$[[x_j,\,x_i],\,x_k]^{e_i} \in D(4,\,\mathfrak{f}\mathfrak{s}\mathfrak{f}),\ \text{provided } j > i. \tag{9}$$

Modulo $\mathfrak{f}^4$, we have

$$\begin{aligned}[][[x_j,\,x_i],\,x_k]^{e_i} - 1 = e_i\{[x_j - 1)(x_i - 1) - (x_i - 1)(x_j - 1)](x_k - 1)\\ - (x_k - 1)[(x_j - 1)(x_i - 1) - (x_i - 1)(x_j - 1)]\}.\end{aligned} \tag{10}$$

Since $j > i$, both $e_i(x_i - 1)$ and $e_i(x_j - 1)$ lie in $\mathfrak{s}\mathbb{Z}[F] + \mathfrak{f}^2$, and consequently (9) holds. Next, consider $[[x_j,\,x_i[,\,x_j],\ j > i$, and let e be an integer such that $e_j|e,\ e_i|2e$. Then, modulo $\mathfrak{f}^4$, we have

$$\begin{aligned}&[[x_j,\,x_i],\,x_j]^e - 1 =\\ &\quad e\{[x_j - 1)(x_i - 1) - (x_i - 1)(x_j - 1)](x_j - 1)\\ &\qquad - (x_j - 1)[(x_j - 1)(x_i - 1) - (x_i - 1)(x_j - 1)]\}.\end{aligned} \tag{11}$$

Given $e_i|2e$ and $e_j|e$, it follows that the above expression lies in $\mathfrak{f}^4 + \mathfrak{f}\mathfrak{s}\mathfrak{f}$.

Let X be the subgroup of F generated by the elements of the type $[[x_j, x_i], x_k]^{e_i}$, $i, j, k \in \{1, \dots, m\}$, $j > i$, and $[[x_j, x_i], x_j]^e$, $j > i$, $e_j|e$, $e_i|2e$.

Recall that the quotient $\gamma_3(F)/\gamma_4(F)$ is a free abelian group with basis

$$\{[[x_j, x_i], x_k]\gamma_4(F) \,|\, 1 \le i, j, k \le m, \; j > i \le k\}.$$

Let $w \in D(4, \mathfrak{f}\mathfrak{s}\mathfrak{f})$. Then, modulo $\gamma_4(F)$, $w = \prod_{i=1}^{m-1} w_i$ with

$$w_i = \prod_{j>i\le k} [[x_j, x_i], x_k]^{a_{ijk}}, \; a_{ijk} \in \mathbb{Z}.$$

By descending induction on i, it follows that $w_i \in D(4, \mathfrak{f}\mathfrak{s}\mathfrak{f})$ for each $i = 1, 2, \dots, m-1$. Now, modulo $\mathfrak{f}\mathfrak{s}\mathfrak{f} + \mathfrak{f}^4$,

$$(12) \quad W_i := w_i - 1 = \sum_{j>i\le k} a_{ijk}\{[(x_j-1)(x_i-1) - (x_i-1)(x_j-1)](x_k-1) - (x_k-1)[(x_j-1)(x_i-1) - (x_i-1)(x_j-1)]\}.$$

Considering the left partial derivative of W_i with respect to x_i, we have

$$(13) \quad {}_{x_i}W_i = \sum_{j>i} a_{iji}[(x_j-1)(x_i-1) - 2(x_i-1)(x_j-1)] - \sum_{j>i,\, k>i} a_{ijk}(x_k-1)(x_j-1) \in \mathfrak{f}^3 + \mathfrak{f}\mathfrak{s}.$$

Considering the right partial derivative of ${}_{x_i}W_i$ with respect to x_i, we have

$$(14) \qquad 2a_{iji}(x_j-1) \in \mathfrak{f}^2 + \mathfrak{s}\mathbb{Z}[F],$$

and, therefore

$$(15) \qquad \sum_{j>i} a_{iji}(x_j-1)(x_i-1) - \sum_{j>i,\, k>i} a_{ijk}(x_k-1)(x_j-1) \in \mathfrak{f}^3 + \mathfrak{f}\mathfrak{s}.$$

Since the second term in (15) does not contain x_i, it follows that

$$(16) \qquad \sum_{j>i} a_{iji}(x_j-1)(x_i-1) \in \mathfrak{f}^3 + \mathfrak{f}\mathfrak{s}$$

and

$$(17) \qquad \sum_{j>i,\, k>i} a_{ijk}(x_k-1)(x_j-1) \in \mathfrak{f}^3 + \mathfrak{f}\mathfrak{s}.$$

From the inclusion (16), it follows that $\sum_{j>i} a_{iji}(x_i-1) \in \mathfrak{f}^2 + \mathfrak{s}\mathbb{Z}[F]$, and consequently

$$(18) \qquad e_i | a_{iji} \text{ for all } j > i.$$

It thus follows that

$$(19) \qquad [[x_j, x_i], x_i]^{a_{iji}} \in X\gamma_4(F).$$

On taking right partial derivative with respect to x_k, $k > i$, the inclusion (17) shows that we have

$$\sum_{j>i} a_{ijk}(x_j - 1) \in \mathfrak{f}^2 + \mathfrak{s}\mathbb{Z}[F], \tag{20}$$

implying that

$$e_j | a_{ijk} \text{ for } j > i,\, k > i. \tag{21}$$

Plugging the foregoing divisibility conditions, obtained so far, back in (12), we see that

$$\begin{aligned}\sum_{j>i<k,\, j\neq k} (a_{ijk} + a_{ikj})[(x_j - 1)(x_i - 1)(x_k - 1) + (x_k - 1)(x_i - 1)(x_j - 1)] \\ + \sum_{j>i} 2a_{ijj}[(x_j - 1)(x_i - 1)(x_j - 1)] \in \mathfrak{f}^4 + \mathfrak{f}\mathfrak{s}\mathfrak{f}.\end{aligned} \tag{22}$$

Considering the left partial derivative with respect to x_k, we have

$$\sum_{j>i} (a_{ijk} + a_{ikj})(x_j - 1)(x_i - 1) \in \mathfrak{f}^3 + \mathfrak{f}\mathfrak{s}. \tag{23}$$

It thus follows that

$$e_i | (a_{ijk} + a_{ikj}) \text{ for all } j > i,\, k > i. \tag{24}$$

The inclusion (22) thus reduces to

$$\sum_{j>i} 2a_{ijj}(x_j - 1)(x_i - 1)(x_j - 1) \in \mathfrak{f}^4 + \mathfrak{f}\mathfrak{s}\mathfrak{f}, \tag{25}$$

implying that

$$e_i | 2a_{ijj} \text{ for all } j > i. \tag{26}$$

Consequently

$$[[x_j,\, x_i],\, x_j]^{a_{ijj}} \in X\gamma_4(F). \tag{27}$$

For $j > i \leq k,\ j \neq k$, we have $e_i | (a_{ijk} + a_{ikj})$, therefore, modulo $X\gamma_4(F)$,

$$\begin{aligned}&[[x_j,\, x_i],\, x_k]^{a_{ijk}}.[[x_k,\, x_i],\, x_j]^{a_{ikj}} \\ &\quad = [[x_j,\, x_i],\, x_k]^{a_{ijk}}.[[x_k,\, x_i],\, x_j]^{-a_{ijk}} \\ &\quad\quad = [[x_k,\, x_j],\, x_i]^{-a_{ijk}} = 1,\end{aligned} \tag{28}$$

provided $k > j$. Here the first equality holds because $e_i | (a_{ijk} + a_{ikj})$, and $[[x_k, x_i], x_j]^{e_i} \in X\gamma_4(F)$, while the second equality holds in view of Hall-Witt identity, and the last equality holds because $e_j | a_{ijk}$. Similar argument shows that the same conclusion holds if $j > k$.

The foregoing analysis shows that $w_i \in X\gamma_4(F)$, and thus the proof of (i) is complete.

The proofs of (ii)-(iv) are similar and so we omit the details. □

The preceding theorem identifies a set of generators for $D(4,\, \mathfrak{f}\mathfrak{r}\mathfrak{f})/\gamma_4(F)$ in terms of the basic commutators $[[x_j,\, x_i],\, x_k]$, $j > i, i \leq k$ of weight three. An easy

calculation on these generators shows that another identification can be given as follows:

Theorem 6. *If F is a free group with ordered basis $\{x_1 < \cdots < x_m\}$, $S = \langle x_1^{e_1}, \ldots, x^{e_m}, \gamma_2(F)\rangle$, $e_m|e_{m-1}|\ldots|e_1$, and R is a normal subgroup of F satisfying $R\gamma_2(F) = S$, then the subgroup $D(4, \mathfrak{frf})$ is generated by the set X consisting of $\gamma_4(F)$ and the elements of the type $[[x_j, x_i], x_k]^e$, $j > i$, $1 \le k \le m$ satisfying*

- $e_i \,|\, e$, *if* $k \neq j$
- $e_j \,|\, e$, $e_i \,|\, 2e$, k=j

Clearly the subgroup $[[R, R], F] \subset 1 + \mathfrak{frf}$, and is generated, modulo $\gamma_4(F)$, by the set Y consisting of the elements

$$[[x_j, x_i], x_k]^{e_ie_j},\ j > i,\ 1 \le k \le m.$$

Hence we have

Corollary 7. $D(4, \mathfrak{frf})/[[R, R], F]\gamma_4(F) \cong \langle X\rangle/\langle Y\rangle\gamma_4(F)$.

Theorem 8. *The subgroup $D(4, \mathfrak{r}^2\mathfrak{f})$ is generated, modulo $\gamma_4(F)$, by the elements $[[x_j, x_i], x_k]^{\ell_{ijk}}$, $1 \le i, j, k \le m$, $j > i \le k$, where*

$$\ell_{ijk} = \begin{cases} \mathrm{lcm}(e_ie_j,\, e_ie_k,\, e_je_k), & k \neq i \\ e_ie_j, & k = i \end{cases}.$$

Proof. As in the proof of the previous result, we have

$$D(4, \mathfrak{r}^2\mathfrak{f}) = D(4, \mathfrak{s}^2\mathfrak{f}) \le \gamma_3(F).$$

It is easy to see that the triple commutators occurring in the statement of the theorem lie in $D(4, \mathfrak{s}^2\mathfrak{f})$. Further, if $w \in D(4, \mathfrak{s}^2\mathfrak{f})$, then

$$w = \prod_{i=1}^{m-1} w_i, \quad w_i = [[x_j, x_i], x_k]^{a_{ijk}},\ j > i \le k,\ w_i \in D(4, \mathfrak{s}^2\mathfrak{f}).$$

Modulo $\mathfrak{s}^2\mathfrak{f} + \mathfrak{f}^4$, we have

$$(29)\quad w_i - 1 = \sum_{j>i\le k} a_{ijk}\{[(x_j-1)(x_i-1) - (x_i-1)(x_j-1)](x_k-1) \\ - (x_k-1)[(x_j-1)(x_i-1) - (x_i-1)(x_j-1)]\}.$$

Considering the left partial derivative with respect to x_i, we have

$$(30)\quad \sum_{j>i} a_{iji}\{(x_j-1)(x_i-1) - 2(x_i-1)(x_j-1)\} \\ - \sum_{j>i<k} a_{ijk}(x_k-1)(x_j-1) \in \mathfrak{s}^2 + \mathfrak{f}^3.$$

Observe that $\mathfrak{f}^2/\mathfrak{f}^3$ is a free abelian group with basis

$$(x_i-1)(x_j-1)+\mathfrak{f}^3,\ 1\le i,\, j\le m,$$

and $\mathfrak{s}^2+\mathfrak{f}^3$ is generated by the elements

$$e_ie_j(x_i-1)(x_j-1)+\mathfrak{f}^3, 1\le i,\, j\le m.$$

Thus the inclusion (30) yields

$$e_ie_j|a_{iji},\quad e_je_k|a_{ijk} \text{ for } j>i<k, \tag{31}$$

and in turn

$$\sum_{j>i<k} a_{ijk}(x_k-1)(x_j-1)\in \mathfrak{s}^2+\mathfrak{f}^3. \tag{32}$$

The above inclusion then yields

$$e_je_k|a_{ijk} \text{ for } j>i<k. \tag{33}$$

The congruence (29) now implies that

$$\sum_{j>i<k} a_{ijk}\{[(x_j-1)(x_i-1)-(x_i-1)(x_j-1)](x_k-1) + (x_k-1)(x_i-1)(x_j-1)]\}\in \mathfrak{s}^2\mathfrak{f}+\mathfrak{f}^4. \tag{34}$$

Considering the left partial derivative with respect to x_j, we obtain

$$a_{ijj}[(x_j-1)(x_i-1)-(x_i-1)(x_j-1)]+\sum_{j>i<k} a_{ijk}(x_k-1)(x_i-1)\in \mathfrak{s}^2+\mathfrak{f}^3. \tag{35}$$

Thus we have

$$e_ie_j|a_{ijj},\ e_ie_k|a_{ijk} \text{ for } j>i<k, \tag{36}$$

forcing (35) to yield

$$e_ie_j|a_{ijk}, \text{for } j>i<k. \tag{37}$$

The various divisibility conditions derived above for the integers a_{ijk} clearly suffice for the claimed assertion. □

Theorem 9. *(i)* $D(3,\, \mathfrak{s}^2)\equiv \gamma_2(S) \mod \gamma_3(F)$. *(ii)* $D(4,\, \mathfrak{s}^2\mathbb{Z}[F])\equiv\gamma_2(S) \mod \gamma_4(F)$.

Proof. Clearly $\gamma_2(S)\le 1+\mathfrak{s}^2$.

(i) Let $w\in D(3,\, \mathfrak{s}^2)$. Then, modulo $\gamma_3(F)$, $w=\prod_{j>i}[x_j,\, x_i]^{a_{ij}}$, $a_{ij}\in\mathbb{Z}$, with

$$\sum_{j>i}[(x_j-1)(x_i-1)-(x_i-1)(x_j-1)]a_{ij}\in \mathfrak{f}^3+\mathfrak{s}^2.$$

Comparing the right partial derivative with respect to x_i, we have

$$\sum_j (x_j-1)a_{ij}\in \mathfrak{f}^2+e_i\mathfrak{s}.$$

Comparing the right partial derivative with respect to x_j, we conclude that

$$e_ie_j\,|\,a_{ij}.$$

Hence it follows that

$$\prod_{j>i}[x_j,\, x_i]^{a_{ij}}=\prod_{j>i}[x_j^{e_j},\, x_i^{e_i}]^{b_{ij}},\ b_{ij}\in\mathbb{Z},$$

and consequently $w \in \gamma_3(F)\gamma_2(S)$.

(ii) Let $w \in D(4, \mathfrak{s}^2\mathbb{Z}[F])$. By part (i), we have

$$w = w_0 \mod \gamma_2(S)$$

for some $w_0 \in \gamma_3(F)$. We claim that $w_0 \in \gamma_4(F)\gamma_2(S)$. Now, modulo $\gamma_4(F)$,

$$w_0 = \prod_i w_i, \; w_i = \prod_{j>i\leq k} [[x_j, x_i], x_k]^{a_{ijk}}, \; a_{ijk} \in \mathbb{Z},$$

and, as can be seen by descending induction on i, we have, modulo $\mathfrak{f}^4 + \mathfrak{s}^2\mathbb{Z}[F]$,

$$\begin{aligned}(38)\quad w_i - 1 = \sum_{j>i\leq k} a_{ijk}\{&(x_k-1)[(x_j-1)(x_i-1)-(x_i-1)(x_j-1)] \\ &- [(x_j-1)(x_i-1)-(x_i-1)(x_j-1)](x_k-1)\}.\end{aligned}$$

Comparing the right partial derivative with respect to x_i, we have

$$\begin{aligned}(39)\quad \sum_{j>i} a_{iji}&[(x_j-1)(x_i-1)-(x_i-1)(x_j-1)] \\ &+ \sum_{j,\,k}(x_j - 1(x_k-1)a_{ijk} \in \mathfrak{f}^3 + t(x_i, e_i)\mathfrak{s}\mathbb{Z}[F] + \mathfrak{f}\mathfrak{s}.\end{aligned}$$

Next, comparing right partial derivative with respect to x_j, we have

$$a_{iji}(x_i-1) + \sum_k (x_k-1)a_{ijk} \in \mathfrak{f}^2 + e_i t(x_j, e_j)\mathbb{Z}[F] + e_i\mathfrak{f} + \mathfrak{s}\mathbb{Z}[F].$$

Thus it follows that

$$(40)\qquad e_k \,|\, a_{ijk}, \; k \neq i,$$

The condition (40) implies that

$$w_i = w_{i1}w_{i2}, \; w_{i1} = \prod_{j>i}[[x_j, x_i], x_i]^{a_{iji}}, \; w_{i2} = \prod_{j>i<k}[[x_j, x_i], x_k]^{a_{ijl}}$$

with $w_{i2} \in \gamma_2(S)\gamma_4(F)$. Consequently, we have

$$w_{i1} - 1 \in \mathfrak{f}^4 + \mathfrak{s}^2\mathbb{Z}[F],$$

i.e.,

$$\begin{aligned}(41)\quad \sum_{j>i} a_{iji}\{&(x_i-1)[(x_j-1)(x_i-1)-(x_i-1)(x_j-1)] \\ &- [(x_j-1)(x_i-1)-(x_i-1)(x_j-1)](x_i-1)\} \in \mathfrak{f}^4 + \mathfrak{s}^2\mathbb{Z}[F].\end{aligned}$$

Comparing right partial derivative with respect to x_j, we have

$$a_{iji}(x_i-1)^2 \in \mathfrak{f}^3 + t(x_j, e_j)\mathfrak{s} + \mathfrak{f}\mathfrak{s}.$$

Next comparing right partial derivative with respect to x_i, we have

$$(x_i-1)a_{iji} \in \mathfrak{f}^2 + e_j t(x_i, e_j)\mathbb{Z}[F] + e_i\mathfrak{f} + \mathfrak{s}\mathbb{Z}[F].$$

Thus it follows that $e_i \,|\, a_{iji}$. Hence $w_{i1} \in \gamma_2(S)\gamma_4(F)$. $\square$

On observing that $\gamma_2(S) \leq (\gamma_2(R)\gamma_3(F))\cap(R\gamma_4(F))$, we have the following result of which (ii) answers Narain Gupta's Problem 6.9(a) in [7].

Corollary 10. *(i)* $D(3,\, \mathfrak{r}^2) = \gamma_2(R)\gamma_3(F)$ *and* *(ii)* $D(4,\, \mathfrak{r}^2\mathbb{Z}[F]) \leq R\gamma_4(F)$.

If $\mathcal{A}$ is a nilpotent ring of exponent $n+1$, i.e., $\mathcal{A}^{n+1} = (0)$, then under the circle operation

$$a \circ b = a + b + ab,\ a,\, b \in \mathcal{A},$$

we have a group $(\mathcal{A},\, \circ)$, called the *circle group* of $\mathcal{A}$. Suppose we have a homomorphism $\theta : G \to (\mathcal{A},\, \circ)$. Then we can extend θ, by linearity, to a ring homomorphism

$$\theta^* : \mathbb{Z}[G] \to R.$$

It is then clear that $\theta^*(\mathfrak{g}^i) \leq \mathcal{A}^i$, and, in particular,

$$\theta^*(\mathfrak{g}^{n+1}) = (0).$$

Thus we have the following:

Proposition 11. (*Sandling* [14]) *If a group G embeds in the circle group of a nilpotent ring of exponent $n+1$, then the $(n+1)$th dimension subgroup $D_{n+1}(G) = \{1\}$.*

As a consequence of the above Proposition and the existence of nilpotent groups of class three without dimension property, we note that, *in general, a nilpotent group of class three cannot be embedded in the circle group of a nilpotent ring of exponent four* (see [1]).

In the foregoing discussion, we have identified the subgroups

$$D(4,\, \mathfrak{a}) = F \cap (1 + \mathfrak{a} + \mathfrak{f}^4)$$

for various two-sided ideals $\mathfrak{a} \subset \mathfrak{f}$. In each of these cases $F/D(4,\, \mathfrak{a})$ embeds in the circle group of the nilpotent ring $\mathfrak{f}/(\mathfrak{a} + \mathfrak{f}^4)$ of exponent four under the map

$$fD(4,\, \mathfrak{a}) \mapsto f - 1 + \mathfrak{a} + \mathfrak{f}^4.$$

Thus in each of these cases we are getting a nilpotent group of class three which has the *dimension property*, i.e., a group whose dimension series agrees with its lower central series.

2. Derived functors

Let us recall the construction of derived functors in the sense of Dold-Puppe [4]. For an endofunctor F on the category $\mathfrak{A}$ of abelian groups, the bigraded sequence $L_i(-,\, n),\ i \geq 0,\ n \geq 0$, of derived functors of F is defined by setting

$$L_iF(A,\, n) = \pi_i(FKP_*[n]),\ A \in \mathfrak{A},$$

where $P_*[n]_* \to A$ is a projective resolution of A starting at level n, and K is the Dold-Kan transform, inverse to the Moore normalization functor from simplicial abelian groups to chain complexes. We are concerned only with the case when $n = 0$, and, as such, we drop the index n and write $L_iF(A) := L_i(A,\, 0)$. Our aim

in this section is to explore relationship between generalized dimension subgroups and derived functors.

Recall that, if $\mathrm{SP}^2(A)$ denotes the second symmetric power of the abelian group A, then $A \mapsto \mathrm{SP}^2(A)$ is a non-additive functor of degree 2 on the category $\mathfrak{A}$. As observed in [8], the functor $L_1\,\mathrm{SP}^2$ is related to the generalized dimension subgroup $D(3,\,-)$. Let us give another proof of that relationship.

Theorem 12. *Let G be a group, G_{ab} its abelianization $G/\gamma_2(G)$, and F/R a free presentation of G. Then*

$$L_1\,\mathrm{SP}^2(G_{ab}) \cong D(3,\,\mathfrak{rf})/\gamma_2(R)\gamma_3(F). \tag{42}$$

Proof. Set $\bar{F} = F_{ab},\ \bar{R} = R\gamma_2(F)/\gamma_2(F)$. Then $G_{ab} \cong \bar{F}/\bar{R}$. Let $\{x_i\}_{i\in I}$ be a free basis of F. The standard identifications

$$\frac{\bar{F}\otimes\bar{F}}{\bar{R}\otimes\bar{F}} = \frac{\mathfrak{f}^2}{\mathfrak{rf}+\mathfrak{f}^3}, \quad \frac{\bar{F}\wedge\bar{F}}{\bar{R}\wedge\bar{R}} = \frac{\gamma_2(F)}{\gamma_2(R)\gamma_3(F)}$$

induced by $\bar{x}_i\otimes\bar{x}_j \mapsto (x_i-1)(x_j-1)+\mathfrak{rf}+\mathfrak{f}^3$, and $\bar{x}_i\wedge\bar{x}_j \mapsto [x_i,\,x_j]\gamma_2(R)\gamma_3(F)$ respectively, imply that there is a natural exact sequence

$$0 \to \frac{D(3,\,\mathfrak{rf})}{\gamma_2(R)\gamma_3(F)} \to \frac{\bar{F}\wedge\bar{F}}{\bar{R}\wedge\bar{R}} \to \frac{\bar{F}\otimes\bar{F}}{\bar{R}\otimes\bar{F}} \tag{43}$$

Now recall that the Koszul complex

$$0 \to \bar{R}\wedge\bar{R} \to \bar{R}\otimes\bar{F} \to \mathrm{SP}^2(\bar{F}) \tag{44}$$

defines the object $L\,\mathrm{SP}^2(G_{ab})$ of the derived category of abelian groups; therefore, H_1 of the complex (44) is naturally isomorphic to $L_1\,\mathrm{SP}^2(G_{ab})$ ([3], [11]). The claimed identification (42) follows from the following commutative diagram with exact columns

$$\begin{tikzcd}
\bar{R}\wedge\bar{R} \arrow[r, tail] \arrow[d, tail] & \bar{R}\otimes\bar{F} \arrow[r] \arrow[d, tail] & \mathrm{SP}^2(\bar{F}) \arrow[d, equal] \\
\bar{F}\wedge\bar{F} \arrow[r, tail] \arrow[d, two heads] & \bar{F}\otimes\bar{F} \arrow[r, two heads] \arrow[d, two heads] & \mathrm{SP}^2(\bar{F}) \\
\frac{\bar{F}\wedge\bar{F}}{\bar{R}\wedge\bar{R}} \arrow[r] & \frac{\bar{F}\otimes\bar{F}}{\bar{R}\otimes\bar{F}} &
\end{tikzcd}$$

where the middle sequence is the standard Koszul exact sequence and the lower horizontal map is the map from (43). □

Let us next recall the well-known description (see, for example, MacLane [12]), of the derived functor $L_1\otimes^2 = \mathrm{Tor}^{[2]} : \mathfrak{A} \to \mathfrak{A}$. Given $A \in \mathfrak{A}$, the group $\mathrm{Tor}^2(A)$ is generated by the n-linear expressions $\tau_h(a_1, a_2)$ (where all a_i belong to the subgroup ${}_hA := \{a \in A \,|\, ha = 0\},\ h > 0$), subject to the so-called slide relations

$$\tau_{hk}(a_1,\,a_2) = \tau_h(ka_1,\,a_2) \tag{45}$$

for all i whenever $hka_1 = 0$ and $ha_2 = 0$ and analogous relation, where the roles of $a_1,\ a_2$ are changed.

The natural map $\otimes^2 \to \mathrm{SP}^2$ induces a natural epimorphism

$$L_1 \otimes^2 (A) \to L_1 \mathrm{SP}^2(A) \tag{46}$$

which maps the generator $\tau_h(a_1,\, a_2)$ of $L_1 \otimes^2 (A) \cong \mathrm{Tor}(A,\, A)$ to the generator $\beta_h(a_1,\, a_2)$ of $L_1 \mathrm{SP}^2(A)$ so that the kernel of this map is generated by the elements $\tau_h(a,\, a),\ a \in {}_hA$.

It is shown by Jean in [10] that

$$L_1 \mathrm{SP}^3(A) \simeq (L_1 \mathrm{SP}^2(A) \otimes A)/Jac_S, \tag{47}$$

where Jac_S is the subgroup generated by elements of the form

$$\beta_h(x_1,\, x_2) \otimes x_3 - \beta_h(x_1,\, x_3) \otimes x_2 + \beta_h(x_2,\, x_3) \otimes x_1$$

with $x_i \in {}_hA$.

There is a natural connection between these functors and the homology of Eilenberg-MacLane spaces $K(\Pi,\, n)$ (see, for example, [2], [3]). In particular, there are natural isomorphisms

$$H_5K(A,\, 2) \cong L_1\Gamma^2(A), \quad H_7K(A,\, 2) \cong L_1\Gamma^3(A)$$

where $\Gamma^2,\ \Gamma^3$ are divided square and cube functor respectively. The natural transformations $\mathrm{SP}^2 \to \Gamma^2$, $\mathrm{SP}^3 \to \Gamma^3$ induce, for $A \in \mathfrak{A}$, the following exact sequences ([10], [3]) (which do not split functorially):

$$0 \to L_1 \mathrm{SP}^2(A) \to H_5K(A,\, 2) \to \mathrm{Tor}(A,\, \mathbb{Z}/2\mathbb{Z}) \to 0$$

and

$$0 \to L_1 \mathrm{SP}^3(A) \to H_7K(A,\, 2) \to \mathrm{Tor}_1(A,\, A,\, \mathbb{Z}/2\mathbb{Z}) \oplus \mathrm{Tor}(A,\, \mathbb{Z}/3\mathbb{Z}) \to 0 \tag{48}$$

We are now ready to present our main result on the relation between generalized dimension subgroups and derived functors.

Theorem 13. *If G_{ab} is 2-torsion-free, then*

$$\frac{D(4,\, \mathfrak{f}\mathfrak{r}\mathfrak{f})}{[R,\, R,\, F]\gamma_4(F)} \simeq L_1 \mathrm{SP}^3(G_{ab})$$

and there is a natural short exact sequence

$$\frac{D(4,\, \mathfrak{f}\mathfrak{r}\mathfrak{f})}{[R,\, R,\, F]\gamma_4(F)} \hookrightarrow H_7K(G_{ab},\, 2) \twoheadrightarrow \mathrm{Tor}(G_{ab},\, \mathbb{Z}/3) \tag{49}$$

In particular, if G_{ab} is both 2-torsion-free and 3-torsion-free, then there is a natural isomorphism

$$\frac{D(4,\, \mathfrak{f}\mathfrak{r}\mathfrak{f})}{[R,\, R,\, F]\gamma_4(F)} \cong H_7K(G_{ab},\, 2). \tag{50}$$

Proof. Let us first consider the natural homomorphism

$$q : L_1 \mathrm{SP}^2(G_{ab}) \otimes G_{ab} \to$$
$$Q(F,\, R) := \frac{\langle [[x_j,\, x_i],\, x_k]^{e_i},\ m \geq j > i \geq 1,\ 1 \leq k \leq m\rangle\gamma_4(F)}{[R,\, R,\, F]\gamma_4(F)}$$

defined by

$$q : \beta_h(\bar{x}_1,\, \bar{x}_2) \otimes \bar{x}_3 \mapsto [x_1,\, x_2,\, x_3]^h.[R,\, R,\, F]\gamma_4(F),\ x_1^h,\ x_2^h \in R\gamma_2(F).$$

Clearly, q is an epimorphism. For $x_3 \in F$ such that $x_3^h \in R\gamma_2(F)$, observe that

$$q(\beta_h(\bar{x}_1, \bar{x}_2) \otimes \bar{x}_3 - \beta_h(\bar{x}_1, \bar{x}_3) \otimes \bar{x}_2 + \beta_h(\bar{x}_2, \bar{x}_3) \otimes \bar{x}_1) = [x_1, x_2, x_3]^h [x_1, x_3, x_2]^{-h} [x_2, x_3, x_1]^h.[R, R, F]\gamma_4(F). \tag{51}$$

By Hall-Witt identity, this element is trivial in $Q(F, R)$. By (47), the map q factors through $L_1\,\mathrm{SP}^3(G_{ab})$. That is, we obtain an induced epimorphism

$$q' : L_1\,\mathrm{SP}^3(G_{ab}) \to Q(F, R).$$

The formula for cross-effect of $L_1\,\mathrm{SP}^3$ is given as follows (see [3], [10]): for abelian groups $A,\ B$,

$$L_1\,\mathrm{SP}^3(A \oplus B) = L_1\,\mathrm{SP}^3(A) \oplus L_1\,\mathrm{SP}^3(B) \oplus L_1\,\mathrm{SP}^3(A|B),$$

with a short exact sequence, which splits (unnaturally)

$$0 \to (L_1\,\mathrm{SP}^2(A) \otimes B) \oplus (A \otimes L_1\,\mathrm{SP}^2(B)) \to L_1\,\mathrm{SP}^3(A|B) \to \mathrm{Tor}(\mathrm{SP}^2(A), B) \oplus \mathrm{Tor}(A,\, \mathrm{SP}^2(B)) \to 0. \tag{52}$$

Observe that using the formula for cross-effect of a polynomial functor together with the value of the functor on cyclic groups one can compute the value of the functor on any finitely generated abelian group. Since $L_1\,\mathrm{SP}^2(\mathbb{Z}/l\mathbb{Z}) = L_1\,\mathrm{SP}^3(\mathbb{Z}/l\mathbb{Z}) = 0$ for any $l \geq 0$, we can easily compute $L_1\,\mathrm{SP}^3(G_{ab})$. It turns out that the group $L_1\,\mathrm{SP}^3(G_{ab})$ is freely generated by triples $w(i,\, j,\, k),\ i < j < k$, and $t(i,\, j,\, k)$, $i \leq j < k$, such that the exponent of $w(i, j, k)$ and of $t(i,\, j,\, k)$ is exactly $e(k)$:

$$L_1\,\mathrm{SP}^3(G_{ab}) \simeq \langle w(i,\, j,\, k),\ i < j < k,\ t(i,\, j,\, k),\ i \leq j < k \mid e(k)w(i,\, j,\, k) = e(k)t(i,\, j,\, k) = 0\rangle \tag{53}$$

The map q' then is given as follows:

$$w(i, j, k) \mapsto [x_j, x_i, x_k]^{e_i},$$
$$t(i, j, k) \mapsto [x_k, x_i, x_j]^{e_i}.$$

Observe that, as an abelian group, the group $Q(F,\, R)$ is freely generated by brackets $[x_j,\, x_i,\, x_k]^{e_i},\ i < j$ with relations $([x_j,\, x_i,\, x_k]^{e_i})^{lcm(e_j, e_k)} = 1$. Hence q' is an isomorphism.

Observe that, if G_{ab} is 2-torsion-free, Corollary 7 implies that there is a natural isomorphism

$$Q(F,\, R) \cong \frac{D(4,\, \mathfrak{frf})}{[R,\, R,\, F]\gamma_4(F)}.$$

The sequence (49) and the isomorphism (50) now follow from the isomorphism $Q(F,\, R) \cong L_1\,\mathrm{SP}^3(G_{ab})$ and the sequence (48). □

We point out that, Theorem 16 (v) from the next section implies that, for any free presentation $G \cong F/R$, there is a natural embedding

$$L_1\,\mathrm{SP}^3(G_{ab}) \hookrightarrow \frac{D(4,\, \mathfrak{frf})}{[R,\, R,\, F]\gamma_4(F)}.$$

The proof of 16 (v) is not combinatorial.

For an abelian group A define $\mathfrak{L}_s^3(A)$ to be the abelian group generated by brackets $\{a,\, b,\, c\},\ a,\, b,\, c \in A$ which are additive on each variable, with the following defining relations:

$$\begin{aligned}
&\{a,b,c\} = \{b,a,c\},\\
&\{a,b,c\} + \{c,a,b\} + \{b,c,a\} = 0.
\end{aligned}$$

This construction defines a functor $\mathfrak{L}_s^3 : \mathsf{Ab} \to \mathsf{Ab}$, which we call *the third super Lie functor*. The difference between $\mathfrak{L}_s^3$ and the third super-Lie functor $\mathcal{L}_s^3$ considered in [3] is that, in general, $\{a,a,a\} \neq 0$ in $\mathfrak{L}_s^3$ and one can get the functor $\mathcal{L}_s^3$ as a natural quotient of $\mathfrak{L}_s^3$ by brackets of the type $\{a,a,a\}$, and we have the following commutative diagram:

$$\begin{tikzcd}
 & & A \otimes \mathbb{Z}/3\mathbb{Z} \arrow[d, tail] \\
\mathrm{SP}^3(A) \arrow[r, tail] \arrow[d, tail] & \mathrm{SP}^2(A) \otimes A \arrow[r, two heads] \arrow[d, tail] & \mathfrak{L}_s^3(A) \arrow[d] \\
\Gamma^3(A) \arrow[r, tail] \arrow[d] & \Gamma^2(A) \otimes A \arrow[r, two heads] \arrow[d] & \mathcal{L}_s^3(A) \\
A \otimes \mathbb{Z}/3\mathbb{Z} \oplus (A \otimes A \otimes \mathbb{Z}/2\mathbb{Z}) \arrow[r, two heads] & A \otimes A \otimes \mathbb{Z}/2\mathbb{Z}. &
\end{tikzcd}$$

Theorem 14. *If G is a group and F/R its free presentation, then*

$$\frac{D(4,\, \mathfrak{r}^2\mathfrak{f})}{\gamma_3(R)\gamma_4(F)} \cong L_2\mathfrak{L}_s^3(G_{ab}).$$

Proof. Let us first construct a natural model for the element $L\mathfrak{L}_s^3(G_{ab})$ of the derived category of abelian groups. We have the the following commutative diagram:

$$\begin{tikzcd}
\Lambda^3(\bar R) \arrow[r, tail] \arrow[d, tail] & \Lambda^2(\bar R) \otimes \bar R \arrow[r, two heads] \arrow[d, tail] & \mathcal{L}^3(\bar R) \arrow[d, tail] \\
\Lambda^2(\bar R) \otimes \bar F \arrow[r, tail] \arrow[d] & \Lambda^2(\bar R) \otimes \bar F \oplus (\bar R \otimes \bar R \otimes \bar F) \arrow[r, two heads] \arrow[d] & \bar R \otimes \bar R \otimes \bar F \arrow[d] \\
\bar R \otimes \mathrm{SP}^2(\bar F) \arrow[r, tail] \arrow[d] & \bar R \otimes \mathrm{SP}^2(\bar F) \oplus (\bar F \otimes \bar R \otimes \bar F) \arrow[r, two heads] \arrow[d] & \bar F \otimes \bar R \otimes \bar F \arrow[d] \\
\mathrm{SP}^3(\bar F) \arrow[r, tail] & \mathrm{SP}^2(\bar F) \otimes \bar F \arrow[r, two heads] & \mathfrak{L}_s^3(\bar F)
\end{tikzcd}$$

where the left vertical complex is the Koszul complex representing the element $L\,\mathrm{SP}^3(G_{ab})$, the middle complex is the tensor product of the Koszul complex with $\bar{R} \to \bar{F}$. Also we have the following commutative diagram:

$$\begin{tikzcd}
\mathcal{L}^3(\bar{R}) \arrow[r, tail] \arrow[d, tail] & \bar{R}\otimes\bar{R}\otimes\bar{F} \arrow[r] \arrow[d, tail] & \bar{R}\otimes\bar{F}\otimes\bar{F} \arrow[r] \arrow[d, tail] & \mathfrak{L}^3_s(\bar{F}) \arrow[d, equal] \\
\mathcal{L}^3(\bar{F}) \arrow[r, tail] \arrow[d] & \bar{F}\otimes\bar{F}\otimes\bar{F} \arrow[r] \arrow[d] & \bar{F}\otimes\bar{F}\otimes\bar{F} \arrow[r, two heads] \arrow[d] & \mathfrak{L}^3_s(\bar{F}) \\
\mathcal{L}^3(\bar{F})/\mathcal{L}^3(\bar{R}) \arrow[r] & \frac{\bar{F}\otimes\bar{F}\otimes\bar{F}}{\bar{R}\otimes\bar{R}\otimes\bar{F}} \arrow[r] & \frac{\bar{F}\otimes\bar{F}\otimes\bar{F}}{\bar{F}\otimes\bar{R}\otimes\bar{F}} &
\end{tikzcd}$$

in which the middle horizontal sequence is exact. Since the upper horizontal sequence models $L\mathfrak{L}^3_s(G_{ab})$, the kernel of the lower left horizontal map is $L_2\mathfrak{L}^3_s(G_{ab})$. On invoking the identifications

$$\frac{\mathfrak{f}^3}{\mathfrak{r}^2\mathfrak{f}+\mathfrak{f}^4} = \frac{\bar{F}\otimes\bar{F}\otimes\bar{F}}{\bar{R}\otimes\bar{R}\otimes\bar{F}}, \quad \gamma_3(F)/(\gamma_3(R)\gamma_4(F)) = \mathcal{L}^3(\bar{F})/\mathcal{L}^3(\bar{R}),$$

our assertion follows from the following natural exact sequence

$$0 \to \frac{D(4,\, \mathfrak{r}^2\mathfrak{f})}{\gamma_3(R)\gamma_4(F)} \to \mathcal{L}^3(\bar{F})/\mathcal{L}^3(\bar{R}) \to \frac{\bar{F}\otimes\bar{F}\otimes\bar{F}}{\bar{R}\otimes\bar{R}\otimes\bar{F}}.$$

□

3. Limits

Let $\mathfrak{A}$ be the category of abelian groups. A *representation* of a category $\mathcal{C}$ is a functor $\mathcal{F} : \mathcal{C} \to \mathfrak{A}$. Let $\mathfrak{A}^{\mathcal{C}}$ denote the category of representations of $\mathcal{C}$. The diagonal functor $diag : \mathfrak{A} \to \mathfrak{A}^{\mathcal{C}}$ is the functor that maps an abelian group A to a constant representation $\mathcal{C} \to \mathfrak{A}$, namely, the one which sends all objects from $\mathcal{C}$ to A and all morphisms to id_A. The *limit of a representation* $\varprojlim$ is a left exact functor $\varprojlim : \mathfrak{A}^{\mathcal{C}} \to \mathfrak{A}$. There are different equivalent ways to define this functor, one way, for example, is to define it as the right adjoint functor to the diagonal functor. We refer the reader to the recent paper [9] for the theory of the functor $\varprojlim$ and its derived functors. If $\mathcal{C}$ is a category such that for any two objects C and C' there exists a morphism $f : C \to C'$. then, for a functor, $\mathcal{F} : \mathcal{C} \to \mathfrak{A}$, $\varprojlim \mathcal{F}$ is the largest constant subfunctor of $\mathcal{F}$ and one can perceive $\varprojlim \mathcal{F}$ as the largest subgroup of $\mathcal{F}(C)$ which does not depend on C.

Let $\mathcal{G}$ be the category of groups, and $\mathcal{E}$ the category of free presentations

$$E : \; 1 \to R \to F \to G \to 1$$

of $G \in \mathcal{G}$. Then, every functor $\mathcal{F} : \mathcal{E} \to \mathfrak{A}$ gives rise to a functor $\mathcal{G} \to \mathfrak{A}$, $G \mapsto \varprojlim \mathcal{F}$. It is naturally of interest to examine the limits of such functors. For example, if $\mathcal{F} : \mathcal{E} \to \mathcal{G}$ is a functor satisfying $\mathcal{F}(E) \subseteq \gamma_n(F)$, $n \geq 2$, then we have a functor $\bar{\mathcal{F}} : \mathcal{E} \to \mathfrak{A}$, $E \mapsto \gamma_n(F)/\mathcal{F}(E)\gamma_{n+1}(F)$, and there arises the problem of describing

$\varprojlim \bar{\mathcal{F}}$. We examine in the present section instances of this type. Let us first recall an observation from [5].

Let $\mathcal{C}$ be a category with pairwise coproducts, i.e., having the property that for any two objects $C_1,\ C_2 \in \mathcal{C}$ there exists the coproduct $C_1 \xrightarrow{i_1} C_1 \sqcup C_2 \xleftarrow{i_2} C_2$ in $\mathcal{C}$. Then, for any representation $\mathcal{F} : \mathcal{C} \to \mathfrak{A}$, there is a natural transformation

$$\mathsf{T}_{\mathcal{F},\,C} : \mathcal{F}(C) \oplus \mathcal{F}(C) \longrightarrow \mathcal{F}(C \sqcup C), \tag{54}$$

given by $\mathsf{T}_{\mathcal{F},\,C} = (\mathcal{F}(i_1),\, \mathcal{F}(i_2))$. The representation $\mathcal{F}$ of $\mathcal{C}$ is said to be *monoadditive* (resp. *additive*) if $\mathsf{T}_{\mathcal{F},\,C}$ is a monomorphism (resp. isomorphism) for all $C \in \mathcal{C}$. The following observation gives a useful tool for description of limits (see [5]):

If $\mathcal{F} : \mathcal{C} \to \mathfrak{A}$ is a monoadditive representation, then $\varprojlim \mathcal{F} = 0$.

Theorem 15. *(Quadratic functors)*

$$(i) \quad \varprojlim \frac{\gamma_2(F)}{[R,\,F]\gamma_3(F)} = \Lambda^2(G_{ab})$$

$$(ii) \quad \varprojlim \frac{\gamma_2(F)}{(R \cap \gamma_2(F))\gamma_3(F)} = \gamma_2(G)/\gamma_3(G)$$

$$(iii) \quad \varprojlim \frac{\gamma_2(F)}{\gamma_2(R)\gamma_3(F)} = L_1\,\mathrm{SP}^2(G_{ab})$$

Proof. The cases (i), (ii) are obvious, since the representations in (i), (ii) are constant. To prove the case (iii), consider the exact sequence

$$1 \to \frac{D(3,\,\mathfrak{rf})}{[R,\,R]\gamma_3(F)} \to \frac{\gamma_2(F)}{[R,\ R]\gamma_3(F)} \to \frac{\mathfrak{f}^2}{\mathfrak{rf} + \mathfrak{f}^3}. \tag{55}$$

There is a natural isomorphism

$$\frac{\mathfrak{f}^2}{\mathfrak{rf} + \mathfrak{f}^3} \cong G_{ab} \otimes F_{ab}.$$

The representation $G_{ab} \otimes F_{ab}$ is monoadditive; moreover, it is, in fact, additive. Therefore, its limit is zero. The statement (iii) follows from the identification of the left hand side of (55) with $L_1\,\mathrm{SP}^2(G_{ab})$ (see Theorem 12). $\square$

Let us next recall the definition of the *tensor square for a non-abelian group.* For a group G, the tensor square $G \otimes G$ is the group generated by the symbols $g \otimes h,\ g,\, h \in G$, satisfying the following defining relations:

$$fg \otimes h = (g^{f^{-1}} \otimes h^{f^{-1}})(f \otimes h) \tag{56}$$

$$f \otimes gh = (f \otimes g)(f^{g^{-1}} \otimes h^{g^{-1}}), \tag{57}$$

for all $f,\, g,\, h \in G$.

The *exterior square*

$$G \wedge G := G \otimes G/(\langle g \otimes g \mid g \in G\rangle$$

is naturally isomorphic to $[F, F]/[R, F]$. Since $G_{ab} \wedge G_{ab} = \gamma_2(F)/([R, F]\gamma_3(F))$, there is a natural isomorphism

$$\frac{\gamma_3(F)}{[R, F] \cap \gamma_3(F)} \cong \ker\{G \wedge G \to G_{ab} \wedge G_{ab}\}$$

Theorem 16. *(Cubic functors)*

$$(i) \quad \varprojlim \frac{\gamma_3(F)}{[R, F, F]\gamma_4(F)} = \mathcal{L}^3(G_{ab})$$

$$(ii) \quad \varprojlim \frac{\gamma_3(F)}{(R \cap \gamma_3(F))\gamma_4(F)} = \gamma_3(G)/\gamma_4(G)$$

$$(iii) \quad \varprojlim \frac{\gamma_3(F)}{([R, F] \cap \gamma_3(F))\gamma_4(F)} = \ker\{G/\gamma_3(G) \wedge G/\gamma_3(G) \to G_{ab} \wedge G_{ab}\}$$

$$(iv) \quad \varprojlim \frac{\gamma_3(F)}{\gamma_3(R)\gamma_4(F)} = L_2\mathfrak{L}^3_s(G_{ab})$$

$$(v) \quad \varprojlim \frac{\gamma_3(F)}{[\gamma_2(R), F]\gamma_4(F)} = L_1 \operatorname{SP}^3(G_{ab})$$

$$(vi) \quad \varprojlim \frac{\gamma_3(F)}{[R, \gamma_2(F)]\gamma_4(F)} = 0$$

Proof. The cases (i) - (iii) are obvious, since the representations are constant.

(iv) Consider the exact sequence

$$1 \to \frac{D(4, \mathfrak{r}^2\mathfrak{f})}{\gamma_3(R)\gamma_4(F)} \to \frac{\gamma_3(F)}{\gamma_3(R)\gamma_4(F)} \to \frac{\mathfrak{f}^3}{\mathfrak{r}^2\mathfrak{f} + \mathfrak{f}^4}$$

The right hand representation lies in the short exact sequence

$$0 \to \bar{R} \otimes \bar{R} \otimes F_{ab} \to F_{ab}^{\otimes 3} \to \frac{\mathfrak{f}^3}{\mathfrak{r}^2\mathfrak{f} + \mathfrak{f}^4} \to 0,$$

where $\bar{R} = R/R \cap [F, F]$. By Proposition 3.7 in [9], $\varprojlim{}^i \bar{R} \otimes \bar{R} \otimes F_{ab} = \varprojlim{}^i F_{ab}^{\otimes 3} = 0,\ i \geq 0$. Hence, $\varprojlim \frac{\mathfrak{f}^3}{\mathfrak{r}^2\mathfrak{f}+\mathfrak{f}^4} = 0$. Consequently, the statement (iv) follows from Theorem 14.

(v) Consider the following commutative diagram with exact rows and columns

$$\begin{tikzcd}
 & & & K \arrow[d, tail] \\
 & & \Lambda^2(\bar{R}) \otimes \bar{F} \arrow[r, equal] \arrow[d, tail] & \Lambda^2(\bar{R}) \otimes \bar{F} \arrow[d] \\
\Lambda^3(\bar{R}) \arrow[d, tail] & \Lambda^3(\bar{F}) \arrow[r, tail] \arrow[d, equal] & \Lambda^2(\bar{F}) \otimes \bar{F} \arrow[r, two heads] \arrow[d] & \mathcal{L}^3(\bar{F}) \arrow[d, two heads] \\
K \arrow[r, tail] \arrow[d, two heads] & \Lambda^3(\bar{F}) \arrow[r] & \frac{\Lambda^2(\bar{F})}{\Lambda^2(\bar{R})} \otimes \bar{F} \arrow[r, two heads] & \frac{\gamma_3(F)}{[R,R,F]\gamma_4(F)} \\
L_2\,\mathrm{SP}^3(G_{ab}) & & &
\end{tikzcd}$$

where $\bar{F} = F_{ab},\ \bar{R} = R/R\cap\gamma_2(F)$ and $K = \ker\{\Lambda^2(\bar{R})\otimes F_{ab} \to \mathcal{L}^3(F_{ab})\}$. It follows from Prop. 3.7 in [9], that

$$\varprojlim \frac{\gamma_3(F)}{[\gamma_2(R), F]\gamma_4(F)} = \varprojlim{}^2\ K = \varprojlim{}^2\ \Lambda^3(\bar{R}).$$

The arguments similar to those given in the proof of Theorem 8.1 in [9] imply that

$$\varprojlim{}^2\ \Lambda^3(\bar{R}) = L_1\,\mathrm{SP}^3(G_{ab})$$

and thus the asserted statement follows.

(vi) Observe that $[R,\, \gamma_2(F)]\gamma_4(F) = [R\gamma_2(F),\, \gamma_2(F)]\gamma_4(F)$. Setting $S = R\gamma_2(F)$, there is an equality

$$[\gamma_2(F),\, S]\gamma_4(F) = (\gamma_2(S) \cap \gamma_3(F))\gamma_4(F).$$

We have the following commutative diagram with exact rows and columns:

$$\begin{tikzcd}
 & \frac{D(4,\mathfrak{fs})}{\gamma_2(S)\gamma_4(F)} \arrow[r, tail, "\theta"] \arrow[d, tail] & \frac{D(3,\mathfrak{fs})}{\gamma_2(S)\gamma_3(F)} \arrow[d, tail] \\
\frac{\gamma_3(F)}{[S,\gamma_2(F)]\gamma_4(F)} \arrow[r, tail] \arrow[d] & \frac{\gamma_2(F)}{\gamma_2(S)\gamma_4(F)} \arrow[r, two heads] \arrow[d] & \frac{\gamma_2(F)}{\gamma_2(S)\gamma_3(F)} \arrow[d] \\
\frac{\mathfrak{f}^3}{\mathfrak{fs}\cap\mathfrak{f}^3+\mathfrak{f}^4} \arrow[r, tail] & \frac{\mathfrak{f}^2}{\mathfrak{fs}+\mathfrak{f}^4} \arrow[r, two heads] & \frac{\mathfrak{f}^2}{\mathfrak{fs}+\mathfrak{f}^3}
\end{tikzcd}$$

Since representations $\mathfrak{f}^2/(\mathfrak{fs}+f^i),\ i=3,4$ are monoadditive,

$$\varprojlim \frac{\mathfrak{f}^2}{\mathfrak{fs}+\mathfrak{f}^3} = \varprojlim \frac{\mathfrak{f}^2}{\mathfrak{fs}+\mathfrak{f}^4} = 0.$$

Hence there is the following commutative square of monomorphisms

$$\begin{tikzcd}
\varprojlim \frac{D(4,\,\mathfrak{fs})}{\gamma_2(S)\gamma_4(F)} \arrow[r, tail, "\varprojlim\ \theta"] \arrow[d, equal] & \frac{D(3,\,\mathfrak{fs})}{\gamma_2(S)\gamma_3(F)} \arrow[d, equal] \\
\varprojlim \frac{\gamma_2(F)}{\gamma_2(S)\gamma_4(F)} \arrow[r] & \varprojlim \frac{\gamma_2(F)}{\gamma_2(S)\gamma_3(F)}.
\end{tikzcd}$$

The middle horizontal exact sequence implies the following exact sequence of limits:

$$1 \to \varprojlim \frac{\gamma_3(F)}{[R,\,\gamma_2(F)]\gamma_4(F)} \to \varprojlim \frac{\gamma_2(F)}{\gamma_2(S)\gamma_4(F)} \to \varprojlim \frac{\gamma_2(F)}{\gamma_2(S)\gamma_3(F)}$$

Since the right hand map is a monomorphism, the stated claim follows. $\square$

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